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Alternative structures and bi-Hamiltonian systems on a Hilbert space

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Abstract

We discuss transformations generated by dynamical quantum systems which are bi-unitary, i.e. unitary with respect to a pair of Hermitian structures on an infinite-dimensional complex Hilbert space. We introduce the notion of Hermitian structures in generic relative position. We provide a few necessary and sufficient conditions for two Hermitian structures to be in generic relative position to better illustrate the relevance of this notion. The group of bi-unitary transformations is considered in both the generic and the non-generic case. Finally, we generalize the analysis to real Hilbert spaces and extend to infinite dimensions results already available in the framework of finite-dimensional linear bi-Hamiltonian systems.

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1. Introduction

The general structures ruling the evolution of classical and quantum systems are not essentially different. For instance both systems are Hamiltonian vector fields and both are derivations on the Lie algebra of observables with respect to the Poisson bracket and the commutator bracket respectively. Besides, in some appropriate limit, quantum mechanics should reproduce classical mechanics [1]. So the question arises of which alternative quantum descriptions for a given quantum system would reproduce the alternative classical descriptions of Hamiltonian systems.These systems are usually known as bi-Hamiltonian systems. Completely integrable systems are often associated with alternative compatible Poisson structures. We recall that by compatibility is usually understood that any combination, with real coefficients, of the two Poisson brackets satisfies the Jacobi identity. In this respect, we should remark that while on a vector space the imaginary part of the Hermitian structures, i.e. constant symplectic structures, are always mutually compatible, this is not true for the full Hermitian structures. In this case the compatibility of the complex structures gives non-trivial conditions even in the vector space situation. As a matter of fact the complex structure, related to the indetermination relation, plays no role in the classical limit of quantum mechanics [2].

In the study of bi-Hamiltonian systems one usually starts with a given dynamics and looks for alternative Hamiltonian descriptions (see a partial list of references for classical [3] and for quantum [4] systems).

In this paper we deal with a kind of converse problem [5], i.e. we start with two Hermitian structures on a complex Hilbert space and look for all dynamical quantum evolutions which turn out to be bi-unitary with respect to them. This study generalizes our previous results on finite-dimensional bi-Hamiltonian systems in [6] to the infinite-dimensional case.

This paper is organized as follows. In section 2, we consider two Hermitian structures on a finite-dimensional Hilbert space and show the equivalence of the following three properties for the Hermitian positive operator G which connects them: the non-degeneracy, the cyclicity and the genericity. A short description of a bi-unitary group is also given. In section 3, we introduce the infinite-dimensional case recalling the direct integral decomposition of a Hilbert space with respect to a commutative ring of operators, which is a suitable mathematical tool to deal with such a situation [7]. In section 4, we extend to the infinite-dimensional Hilbert spaces the analysis drawn in section 2. In particular, we prove that the component spaces in the decomposition are one dimensional if and only if the Hermitian structures are in relative generic position. Also, we show that this happens if and only if the operator G connecting the two Hermitian structures is cyclic. This allows us to conclude that all the quantum systems, which are bi-unitary with respect to two Hermitian structures in generic relative position, commute among themselves. Moreover, the bi-unitary group is explicitly exhibited both in the generic and the non-generic case. In section 5, the analysis starts from different complexifications of a real Hilbert space to discuss the previous results in the light of the notion of compatible triples. [6, 8]. In section 6 we discuss a simple example of some physical interest and finally, in the last section, we draw a few conclusions.

2. Bi-unitary group on a finite-dimensional space

In quantum mechanics the Hilbert space $\mathcal{H}$ is given as a *complex* vector space, because the complex structure enters directly the Schrödinger equation of motion.

Denoting with $h_1(.,.)$ and $h_2(.,.)$ two Hermitian structures given on $\mathcal{H}$ (both linear, for instance, in the second factor), we search for the group of transformations which leave both h_1 and h_2 invariant, that is the bi-unitary transformation group.

By using the Riesz theorem a bounded, positive operator G may be defined, which is self-adjoint both with respect to h_1 and h_2, as

$$h_2(x, y) = h_1(Gx, y), \qquad \forall x, y \in \mathcal{H}. \tag{1}$$

Moreover, any bi-unitary transformation U must commute with G. Indeed

$$h_1(x, U^\dagger GUy) = h_1(Ux, GUy) = h_2(Ux, Uy) = h_2(x, y) = h_1(Gx, y) = h_1(x, Gy)$$

and from this

$$U^\dagger GU = G \quad \Leftrightarrow \quad [G, U] = 0. \tag{2}$$

Therefore, the group of bi-unitary transformations is contained in the commutant G' of the operator G.

To visualize these transformations, let us consider the bi-unitary group of transformations when $\mathcal{H}$ is finite dimensional. In this case G is diagonalizable and the two Hermitian structures result proportional in each eigenspace of G *via* the eigenvalue. Then the group of bi-unitary transformations is given by

$$U(n_1) \times U(n_2) \times \cdots \times U(n_m), \qquad n_1 + n_2 + \cdots + n_m = n = \dim \mathcal{H}, \tag{3}$$

where n_k denotes the degeneracy of the kth eigenvalue of G.

The picture should be clear now. Each Hermitian structure on $\mathcal{H}$ defines a different realization of the unitary group as a group of transformations. The intersection of these two groups identifies the group of bi-unitary transformations.

In finite-dimensional complex Hilbert spaces the following definition can be introduced [6]:

Definition 1. *Two Hermitian forms are said to be in generic relative position when the eigenvalues of G are non-degenerate.*

Then, if h_1 and h_2 are in generic position, the group of bi-unitary transformations becomes

$$\underbrace{U(1) \times U(1) \times \cdots \times U(1)}_{n \text{ factors}}.$$

In other words, this means that G generates a complete set of commuting observables.

Now, recalling that an operator is cyclic when a vector x_0 exists such that the set $\{x_0, Gx_0, \ldots, G^{n-1}x_0\}$ spans the whole n-dimensional Hilbert space, we show that

Proposition 1. *Two Hermitian forms are in generic relative position if and only if their connecting operator G is cyclic.*

Proof. The non-singular operator G has a discrete spectrum and is diagonalizable so, when h_1 and h_2 are in generic position, G admits n distinct eigenvalues λ_k. Let now $\{e_k\}$ be the eigenvector basis of G and $\{\mu^k\}$ an n-tuple of nonzero complex numbers. The vector

$$x_0 = \sum_k \mu^k e_k \tag{4}$$

is a cyclic vector for G. In fact one obtains

$$G^m x_0 = \sum_k \mu^k \lambda_k^m e_k, \qquad m = 0, 1, \ldots, n-1. \tag{5}$$

The vectors $\{G^m x_0\}$ are linearly independent because the determinant of their components is given by

$$\left(\prod_k \mu^k\right) V(\lambda_1, \ldots, \lambda_n), \tag{6}$$

where V denotes the Vandermonde determinant which is different from zero when all the eigenvalues λ_k are distinct. The converse is also true. □

This shows that definition (1) may be equivalently formulated as

Definition 2. *Two Hermitian forms are said to be in generic relative position when their connecting operator G is cyclic.*

The genericity condition can also be restated in a purely algebraic form as follows:

Definition 3. *Two Hermitian forms are said to be in generic relative position when $G'' = G'$, i.e. when the bi-commutant of G coincides with the commutant of G.*

Equivalence of definitions (3) and (1) is apparent.

The last two equivalent properties of G are readily suitable for an extension of the genericity condition to the infinite-dimensional case while, at a first glance, the definition based on non-degeneracy of the spectrum of G looks hardly generalizable.

3. Decomposing an infinite-dimensional Hilbert space

Now we deal with the infinite-dimensional case, when the connecting operator G may have a point part and a continuum part in its spectrum.

As regards to the point part, the bi-unitary group is $U(n_1) \times \cdots \times U(n_k) \times \ldots$, where now n_k may also be ∞. When G admits a continuum spectrum, the characterization of the bi-unitary group is more involved and suitable mathematical tools are needed from the spectral theory of operators and the theory of rings of operators on Hilbert spaces.

We recall that each commutative (weakly closed) ring of operators C in a Hilbert space, containing the identity, corresponds to a direct integral of Hilbert spaces.

The following theorems [7] are useful:

Theorem 1. *To each direct integral of Hilbert spaces with respect to a measure σ on a real interval Δ:*

$$\mathcal{H} = \int_\Delta H_\lambda \, \mathrm{d}\sigma(\lambda),$$

there corresponds a commutative weakly closed ring $C = L^\infty_\sigma(\Delta)$, where to each $\varphi \in L^\infty_\sigma(\Delta)$ there corresponds the operator $L_\varphi : (L_\varphi \xi) = \varphi(\lambda)\xi_\lambda$ with $\xi \in \mathcal{H}$, $\xi_\lambda \in H_\lambda$ and $\|L_\varphi\| = \|\varphi\|_\infty$. Vice versa:

Theorem 2. *To each commutative weakly closed ring C of operators in a Hilbert space $\mathcal{H}$ there corresponds a decomposition of $\mathcal{H}$ into a direct integral, for which C is the set of operators of the form L_φ, $\varphi \in L^\infty$.*

To apply the previous theorems to the ring $R(G)$ generated by the connecting operator G, we preliminarily remark that

Proposition 2. *The weakly closed commutative ring $R(G)$ generated by the connecting operator G contains the identity.*

Proof. Let E_0 be the principal identity of G in the ring of all bounded operators $\mathcal{B}(\mathcal{H})$: by definition E_0 is the projection operator on the orthogonal complement of the set $\ker G$.

We recall [7] that the minimal weakly closed ring $R(G)$ containing G contains only those elements $A \in G''$ which satisfy, like G, the following condition:

$$E_0 A = A E_0 = A. \tag{7}$$

Now the positiveness of the operator G ensures that $\ker G = 0$. This implies that $E_0 = \mathbf{1} \in R(G)$. □

Then, by theorem (2), the ring $R(G)$ induces a decomposition of the Hilbert space $\mathcal{H}$ into the direct integral

$$\mathcal{H} = \int_\Delta H_\lambda \, \mathrm{d}\sigma(\lambda), \tag{8}$$

where $\Delta = [a, b]$ contains the entire spectrum of the positive self-adjoint operator G. The measure $\sigma(\lambda)$ in equation (8) is obtained by the spectral family $\{P_G(\lambda)\}$ of G and cyclic vectors in the usual way [7].

We remark that it results in $R(G) \equiv G''$. Therefore G'' is commutative.

Now every operator A from the commutant G' is representable in the form of a direct integral of operators

$$A = \int_{\Delta} A(\lambda)\, \mathrm{d}\sigma(\lambda), \tag{9}$$

where $A(\lambda)$ is a bounded operator in H_λ, for almost every $\lambda \in \Delta$.

Thus the bi-unitary transformations, as they belong to G', are in general a direct integral of unitary operators $U(\lambda)$ acting on H_λ.

In particular, every operator B of the bi-commutant $G'' = R(G)$ is a multiplication by a number $b(\lambda)$ on H_λ, for almost every λ:

$$B(\lambda) = b(\lambda)1_\lambda. \tag{10}$$

4. Bi-unitary group on an infinite-dimensional Hilbert space

More insight can be gained from a more specific analysis of the direct integral decomposition of $\mathcal{H}$, which can be written as

$$\mathcal{H} = \int_{\Delta} H_\lambda\, \mathrm{d}\sigma(\lambda) = \bigoplus_k \int_{\Delta_k} H_\lambda\, \mathrm{d}\sigma(\lambda) = \bigoplus_k \mathcal{H}_k, \tag{11}$$

where now the spectrum Δ of G is the union of a countable number of measurable sets Δ_k, such that for $\lambda \in \Delta_k$ the spaces H_λ have the same dimension n_k (n_k may be ∞).

The measure $\sigma(\lambda)$ is obtained by the measures $\sigma_k(\lambda)$ *via* the spectral family $\{P_G(\lambda)\}$ of G and cyclic vectors u_k , with $\sigma_k(\lambda) = (P_G(\lambda)u_k, u_k)$.

The dimension n_k of the spaces H_λ is the analogue of the degeneracy of the eigenvalues λ of the point part of the spectrum of G.

According to the decomposition of equation (11), any operator A in the commutant G' is representable as

$$A = \bigoplus_k \int_{\Delta_k} A(\lambda)\, \mathrm{d}\sigma(\lambda). \tag{12}$$

In particular, the connecting operator G is a multiplication by λ on each H_λ, so we get the following result at once:

Proposition 3. *Let two Hermitian structures h_1 and h_2 be given on the Hilbert space $\mathcal{H}$. Then there exists a decomposition of $\mathcal{H}$ into a direct integral of Hilbert spaces H_λ such that in each space H_λ the structures $h_1|_{H_\lambda}$ and $h_2|_{H_\lambda}$ are proportional: $h_2|_{H_\lambda} = \lambda h_1|_{H_\lambda}$.*

Moreover, as G acts like a multiplicative operator on each component space H_λ, the expressions of h_1 and h_2 on $\mathcal{H}$ are

$$\begin{aligned} h_1(x, y) &= \sum_k \int_{\Delta_k} \langle x_\lambda, y_\lambda\rangle_\lambda\, \mathrm{d}\sigma(\lambda) \\ h_2(x, y) &= \sum_k \int_{\Delta_k} \lambda \langle x_\lambda, y_\lambda\rangle_\lambda\, \mathrm{d}\sigma(\lambda), \end{aligned} \tag{13}$$

where $\langle x_\lambda, y_\lambda\rangle_\lambda$ is the inner product on the component H_λ.

As a consequence of proposition (3) and equation (12), the elements U of the bi-unitary group acting on $\mathcal{H}$ have the form

$$U = \bigoplus_k \int_{\Delta_k} U_{n_k}(\lambda)\, \mathrm{d}\sigma(\lambda), \tag{14}$$

where $U_{n_k}(\lambda)$ is an element of the unitary group $U(n_k)$ for each $\lambda \in \Delta_k$.

As regards to the notion of two Hermitian forms in generic position, the following statement [9] holds:

Proposition 4. *Two Hermitian structures h_1 and h_2 are in generic relative position if and only if the component spaces H_λ of the decomposition of $\mathcal{H}$ into a direct integral with respect to $R(G)$ are one dimensional.*

Proof. Let us suppose that two Hermitian forms are given in generic relative position. Then, by definition (3), $R(G) = G'' = G'$, so G' is commutative and any component operator $A(\lambda)$ in equation (12) acts on a one-dimensional component space H_λ, for almost every $\lambda \in \Delta$.

In order to prove the converse, observe that if $R(G) = G'' \neq G'$, then G' is not commutative. So a subset Δ_0 of Δ exists such that $\dim H_\lambda > 1$ for $\lambda \in \Delta_0$. □

This shows the equivalence of definitions (1) and (3) also in the infinite-dimensional case.

Propositions (3) and (4) extend to infinite-dimensional complex Hilbert spaces some results of our previous work [6], so that we can say that all quantum dynamical bi-Hamiltonian systems are pairwise commuting if (and only if) the two Hermitian structures are in generic relative position.

In the generic case, the unitary component operators $U_{n_k}(\lambda)$ in equation (14) reduce to a multiplication by a phase factor $\exp(\mathrm{i}\vartheta(\lambda))$ on H_λ for almost every λ, so that the elements of the bi-unitary group read

$$U = \int_\Delta \mathrm{e}^{\mathrm{i}\vartheta(\lambda)}\,\mathrm{d}\sigma(\lambda). \tag{15}$$

Therefore, in the generic case the group of bi-unitary transformations is parametrized by the σ—measurable real functions ϑ on Δ. This shows that the bi-unitary group may be written as

$$U_\vartheta = \exp(\mathrm{i}\vartheta(G)). \tag{16}$$

Finally, like in the finite-dimensional case, an equivalence may be stated between the genericity condition and the cyclicity of the operator G. In fact, we have

Proposition 5. *Let G be a bounded positive self-adjoint operator in $\mathcal{H}$. Then G is cyclic if and only if $G'' = G'$.*

Proof. Let us suppose $G'' = G'$. Then $R(G) = G'' = G'$ and G' is commutative. Hence the decomposition of the Hilbert space yields one-dimensional component spaces H_λ where G acts as a multiplication by λ in $L_2(\Delta, \sigma)$. Then the vector $x_0 = 1/\lambda$ is a cyclic vector in $L_2(\Delta, \sigma)$, so G is cyclic.

Conversely, let G be cyclic. Then each space H_λ is one dimensional and any operator from G' acts as a multiplication by a number in H_λ. Hence $G' = R(G) = G''$. □

Summarizing, we have shown the equivalence of definitions (1), (2) and (3) in the infinite-dimensional case.

5. Compatible structures on a real infinite-dimensional Hilbert space

In the previous section we have analysed the setting of a complex Hilbert space $\mathcal{H}$ with two Hermitian structures $h_1(.,.)$ and $h_2(.,.)$ and now, to make contact with real linear Hamiltonian mechanics [6] on infinite-dimensional spaces, we analyse the consequences of this on real Hilbert spaces. Besides, the real context displays richer contents and is a more general setting for the analysis of our geometric structures.

We start therefore with a real vector space $\mathcal{H}^{\mathcal{R}}$ (isomorphic to the realification of $\mathcal{H}$). From the two Hermitian structures on the previous complex Hilbert space, $h_1(.,.)$ and $h_2(.,.)$, we get on $\mathcal{H}^{\mathcal{R}}$ two metric tensors g_a and two symplectic forms ω_a *via*

$$g_a(x, y) = \Re h_a(x, y); \qquad \omega_a(x, y) = \Im h_a(x, y), \qquad a = 1, 2.$$

On $\mathcal{H}^{\mathcal{R}}$ the multiplication by the imaginary unit appears as the action of a linear operator J, $J^2 = -1$, which is skew-adjoint with respect to both g.

The structures are related by the equation $\omega_a(x, y) = g_a(Jx, y)$ which defines the *admissible* triples (g_a, ω_a, J).

Then the three linear operators $G^{\mathcal{R}} = g_1^{-1} \circ g_2$, $T = \omega_1^{-1} \circ \omega_2 = -J \circ G^{\mathcal{R}} \circ J$ and J are a set of mutually commuting linear operators, $G^{\mathcal{R}}$ and T being self-adjoint with respect to both metric tensors. We remark, by the way, that T is the recursion operator for symplectic structures.

For instance, to check that $[G^{\mathcal{R}}, J] = 0$, consider the equation $h_2(x, y) = h_1(Gx, y)$ which defines the connecting operator G. Then

$$\begin{aligned} h_1(Gx, y) &= g_1(Gx, y) + \mathrm{i} g_1(JGx, y) = h_2(x, y) \\ &= g_2(x, y) + \mathrm{i} g_2(Jx, y) = g_1(G^{\mathcal{R}}x, y) + \mathrm{i} g_1(G^{\mathcal{R}}Jx, y). \end{aligned}$$

This shows, by equating real and imaginary parts, that $G^{\mathcal{R}} = G$ and $[G, J] = 0$. It is trivial now that $[T, G] = [T, J] = 0$ as well. By definition this means that these two triples are *compatible* [6].

Quantum theory in the usual complex context leads quite naturally to consider identical complex structures in the two triples. In contrast, in the real context it is possible to consider the case of two distinct complex structures J_1, J_2. In other words, on a real Hilbert space $\mathcal{H}^{\mathcal{R}}$ let two admissible triples (g_1, J_1, ω_1) and (g_2, J_2, ω_2) be given which are compatible, that is the commuting operators $\{G, T, J_1, J_2\}$ have the correct bi-Hermiticity properties [8].

Now it is possible to complexify $\mathcal{H}^{\mathcal{R}}$ and to get a complex Hilbert space $\mathcal{H}_1$ with a Hermitian scalar product $\langle ., . \rangle_1$ *via* (g_1, J_1, ω_1). Since by hypothesis the operators $\{G, T, J_2\}$ commute with J_1, they become complex-linear operators on $\mathcal{H}_1$. In particular G becomes a complex-linear bounded positive self-adjoint operator, therefore G acts as a multiplication by λ on the component spaces in the associated direct integral decomposition

$$\mathcal{H} = \int_{\Delta} H_\lambda \, \mathrm{d}\sigma(\lambda). \tag{17}$$

Now J_2 commutes with G, i.e. $J_2 \in G'$, so J_2 is block diagonal on $\mathcal{H}$. In each H_λ, we have $J_2^2(\lambda) = -1_\lambda$ and $J_2^\dagger(\lambda) = -J_2(\lambda)$. Then H_λ splits in two parts corresponding to the eigenvalues $\pm\mathrm{i}$ of $J_2(\lambda) : H_\lambda = H_\lambda^+ \oplus H_\lambda^-$, where on $H_\lambda^+ : J_2 = J_1 = \mathrm{i}$, while on $H_\lambda^- : J_2 = -J_1 = -\mathrm{i}$. The direct integral decomposition becomes

$$\mathcal{H} = \int_{\Delta} H_\lambda^+ \oplus H_\lambda^- \, \mathrm{d}\sigma(\lambda) = \mathcal{H}^+ \oplus \mathcal{H}^- = \int_{\Delta^+} H_\lambda^+ \, \mathrm{d}\sigma(\lambda) \oplus \int_{\Delta^-} H_\lambda^- \, \mathrm{d}\sigma(\lambda), \tag{18}$$

where Δ^+ and Δ^-, subsets of Δ not necessarily disjoint, are support of H_λ^+ and H_λ^- respectively. This completely extends the finite-dimensional result in [6].

At this point we can draw a complete picture: starting from two admissible triples $(g_a, J_a, \omega_a), a = 1, 2$, on $\mathcal{H}^{\mathcal{R}}$ we may construct the corresponding Hermitian structures $h_a = g_a + \mathrm{i}\omega_a$. We stress that h_a is a Hermitian structure on $\mathcal{H}_a$, which is the complexification of $\mathcal{H}^{\mathcal{R}}$ *via* J_a, so that in general h_1 and h_2 are not Hermitian structures on the *same* complex vector space.

When the triples are compatible the decomposition of the space in equation (18) holds, so that $\mathcal{H}^{\mathcal{R}}$ can be decomposed into the direct sum of the spaces $\mathcal{H}_{\mathcal{R}}^+$ and $\mathcal{H}_{\mathcal{R}}^-$ on which

$J_2 = \pm J_1$, respectively. The comparison of h_1 and h_2 requires a fixed complexification of $\mathcal{H}^{\mathcal{R}}$, for instance $\mathcal{H}_1 = \mathcal{H}_1^+ \oplus \mathcal{H}_1^-$. Then, using equations (13) and (18), we can write

$$h_1(x, y) = \int_{\Delta^+} \langle x_\lambda, y_\lambda \rangle_\lambda \, \mathrm{d}\sigma(\lambda) + \int_{\Delta^-} \langle x_\lambda, y_\lambda \rangle_\lambda \, \mathrm{d}\sigma(\lambda), \tag{19}$$

while

$$h_2(x, y) = \int_{\Delta^+} \lambda \langle x_\lambda, y_\lambda \rangle_\lambda \, \mathrm{d}\sigma(\lambda) + \int_{\Delta^-} \lambda \langle y_\lambda, x_\lambda \rangle_\lambda \, \mathrm{d}\sigma(\lambda). \tag{20}$$

It is apparent that h_2 is not a Hermitian structure as it is neither linear nor anti-linear on the whole space $\mathcal{H}_1$.

6. Example: particle in a box, a double case

Consider the operator $G = 1 + X^2$, with X position operator, on $L_2([-\alpha, \alpha], \mathrm{d}x)$. It is Hermitian with spectrum $\Delta = [1, 1 + \alpha^2]$. From the spectral family of X:

$$P(\lambda) f = \chi_{[-\alpha,\lambda]} f, \tag{21}$$

where $\chi_{[-\alpha,\lambda]}$ is the characteristic function of the interval $[-\alpha, \lambda]$, we get the spectral family $P_G(\lambda)$ of G:

$$P_G(\lambda) = P(\sqrt{\lambda - 1}) - P(-\sqrt{\lambda - 1}). \tag{22}$$

In fact, by a simple computation it is immediate to check that P_G is a projection operator:

$$P_G^2 = P_G, \qquad P_G(0) = 0, \qquad P_G(\alpha^2) = 1. \tag{23}$$

Furthermore, write G as

$$G = \int_{[-\alpha,\alpha]} (1 + \lambda^2) \, \mathrm{d}P(\lambda) = \int_{[-\alpha,0]} (1 + \lambda^2) \, \mathrm{d}P(\lambda) + \int_{[0,\alpha]} (1 + \lambda^2) \, \mathrm{d}P(\lambda), \tag{24}$$

and change the variable putting $\lambda = -\sqrt{\mu - 1}$ in the first integral and $\lambda = \sqrt{\mu - 1}$ in the second one. Eventually, the spectral decomposition of G reads

$$G = \int_{[1,1+\alpha^2]} \lambda \, \mathrm{d}P_G(\lambda), \tag{25}$$

where $P_G(\lambda)$ is given by equation (22).

Now G does not have cyclic vectors on the whole $L_2([-\alpha, \alpha], \mathrm{d}x)$, because if f is any vector, $x f(-x)$ is non-zero and orthogonal to all powers $G^n f$. In other words G', which contains both X and the parity operator, is not commutative.

This argument fails on $L_2([0, \alpha], \mathrm{d}x)$, where $\chi_{[0,\alpha]}$ is cyclic. Analogously, $\chi_{[-\alpha,0]}$ is cyclic on $L_2([-\alpha, 0], \mathrm{d}x)$, so we get the splitting in two G-cyclic spaces

$$L_2[-\alpha, \alpha] = L_2[-\alpha, 0] \oplus L_2[0, \alpha]. \tag{26}$$

From P_G and those cyclic vectors we obtain the measure

$$\sigma(\lambda) = (P_G(\lambda)\chi_{[0,\alpha]}, \chi_{[0,\alpha]}) = \sqrt{\lambda - 1} \tag{27}$$

for the decomposition of the Hilbert space

$$\mathcal{H} = \int_{[1,1+\alpha^2]} H_\lambda \, \mathrm{d}\sigma(\lambda), \tag{28}$$

where H_λ is one dimensional for the particle in the $[0, \alpha]$ box while it is bi-dimensional for the $[-\alpha, \alpha]$ box.

The general case of an asymmetric box $[-\alpha, \beta]$ is a direct superposition of the two previous cases, as we have shown in section 4: in fact, assuming $\beta > \alpha$ for instance, the decomposition becomes the direct sum of bi-dimensional spaces for the $[-\alpha, \alpha]$ box plus one-dimensional spaces for the $[\alpha, \beta]$ box.

The bi-unitary transformations U read

$$U = \int_{[1,1+\alpha^2]} e^{i\varphi(\lambda)} \, d\sqrt{\lambda - 1} \tag{29}$$

in the $[0, \alpha]$ box, and

$$U = \int_{[1,1+\alpha^2]} U_2(\lambda) \, d\sqrt{\lambda - 1} \tag{30}$$

in the $[-\alpha, \alpha]$ box. Finally, in the $[-\alpha, \beta]$ box:

$$U = \int_{[1,1+\alpha^2]} U_2(\lambda) \, d\sqrt{\lambda - 1} \oplus \int_{[1+\alpha^2,1+\beta^2]} e^{i\varphi(\lambda)} \, d\sqrt{\lambda - 1}. \tag{31}$$

7. Concluding remarks

In this paper we have shown how to extend to the more realistic case of infinite dimensions the results of our previous paper dealing mainly with finite level quantum systems. Our approach shows, in the framework of quantum systems, how to deal with 'pencils of compatible Hermitian structures' in the same spirit of 'pencils of compatible Poisson structures' [10, 11]. We hope to be able to extend these results to the evolutionary equations for classical and quantum field theories.

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